

A SHARP SOBOLEV–STRICHARTZ ESTIMATE FOR THE WAVE EQUATION

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ABSTRACT. We calculate the sharp constant and characterise the extremal initial data in $\dot{H}^{\frac{3}{4}} \times \dot{H}^{-\frac{1}{4}}$ for the L^4 Sobolev–Strichartz estimate for the wave equation in four space dimensions.

1. INTRODUCTION

For $d \geq 2$ and $s \in [\frac{1}{2}, \frac{d}{2})$ the well-known Sobolev–Strichartz estimate for the one-sided wave propagator states that, for some finite constant $C > 0$,

$$(1) \quad \|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^{d+1})} \leq C\|f\|_{\dot{H}^s(\mathbb{R}^d)}$$

for each f in the homogeneous Sobolev space $\dot{H}^s(\mathbb{R}^d)$, with norm given by $\|f\|_{\dot{H}^s(\mathbb{R}^d)} = \|(-\Delta)^{\frac{s}{2}}f\|_{L^2(\mathbb{R}^d)}$, and where

$$p = \frac{2(d+1)}{d-2s}.$$

The sharp constant in the estimate (1) given by

$$\mathbf{W}(d, s) := \sup_{f \in \dot{H}^s \setminus \{0\}} \frac{\|e^{it\sqrt{-\Delta}}f\|_{L^p(\mathbb{R}^{d+1})}}{\|f\|_{\dot{H}^s(\mathbb{R}^d)}}$$

has attracted attention in recent years; however, to date, the value of $\mathbf{W}(d, s)$ and a full characterisation of extremisers (those f which attain the supremum) has been established only in some rather isolated cases. It is known that, for all admissible (d, s) , an extremiser *exists* (see [5], [7], [14]). Identifying the *exact shape* of such extremisers appears to be a rather difficult problem, with prior results in this direction only available in the cases (d, s) equal to $(2, \frac{1}{2})$ and $(3, \frac{1}{2})$, due to Foschi [8], and the case (d, s) equal to $(5, 1)$ in [3]. In each of these cases, the initial datum f_\star whose Fourier transform is given by

$$\widehat{f_\star}(\xi) = \frac{e^{-|\xi|}}{|\xi|}$$

is extremal; in fact, these works also gave a full characterisation of the extremal data by showing that any extremiser f coincides with f_\star up to the action of a certain group of transformations (which are slightly different when $s = \frac{1}{2}$ and $s = 1$). Based on these results, it is tempting to boldly conjecture that such f are extremisers for all admissible (d, s) . Whilst this is premature, the purpose of this short paper is to add further weight and show that this is indeed the case for $(d, s) = (4, \frac{3}{4})$.

Theorem 1.1. *The one-sided wave propagator satisfies the estimate*

$$(2) \quad \|e^{it\sqrt{-\Delta}}f\|_{L^4(\mathbb{R}^5)} \leq \mathbf{W}(4, \frac{3}{4})\|f\|_{\dot{H}^{\frac{3}{4}}(\mathbb{R}^4)}$$

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with constant

$$\mathbf{W}(4, \frac{3}{4}) = \left(\frac{4}{15\pi^2} \right)^{\frac{1}{4}}.$$

The constant is sharp and is attained if and only if

$$\widehat{f}(\xi) = \frac{e^{a|\xi|+ib\cdot\xi+c}}{|\xi|},$$

where $a, c \in \mathbb{C}$ such that $\operatorname{Re}(a) < 0$, and $b \in \mathbb{R}^d$.

Our proof of Theorem 1.1 relies on a sharp estimate for the one-sided wave propagator from [3]; this is followed by a further argument using spherical harmonics inspired by recent work of Foschi [9] on the sharp Stein–Tomas adjoint Fourier restriction theorem for the sphere \mathbb{S}^2 in \mathbb{R}^3 . We also show that such an approach may be used to recover in a new manner the characterisation of extremisers in [3] for the case $(d, s) = (5, 1)$.

For the full solution of the wave equation, we may deduce the following sharp Sobolev–Strichartz estimate and characterisation of extremal initial data.

Corollary 1.2. *The solution of the wave equation $\partial_t u = \Delta u$ on $\mathbb{R}^4 \times \mathbb{R}$ with initial data $(u(0), \partial_t u(0))$ satisfies*

$$\|u\|_{L^4(\mathbb{R}^5)} \leq \left(\frac{1}{10\pi^2} \right)^{\frac{1}{4}} \left(\|u(0)\|_{\dot{H}^{\frac{3}{4}}(\mathbb{R}^4)}^2 + \|\partial_t u(0)\|_{\dot{H}^{-\frac{1}{4}}(\mathbb{R}^4)}^2 \right)^{\frac{1}{2}}$$

and the constant is sharp. Furthermore, the initial data given by

$$(u(0), \partial_t u(0)) = (0, (1 + |x|^2)^{-\frac{5}{2}}),$$

is extremal and generates the set of all extremal initial data under the action of the group generated by the transformations:

- space-time translations $u(t, x) \rightarrow u(t + t_0, x + x_0)$ with $(t_0, x_0) \in \mathbb{R}^{d+1}$;
- parabolic dilations $u(t, x) \rightarrow u(\mu^2 t, \mu x)$ with $\mu > 0$;
- change of scale $u(t, x) \rightarrow \mu u(t, x)$ with $\mu > 0$;
- phase shift $u(t, x) \rightarrow e^{i\theta} u(t, x)$ with $\theta \in \mathbb{R}$.

Our results here also fit into a broader collection of recent papers on sharp Sobolev–Strichartz estimates for dispersive propagators, where, broadly speaking, the question of existence of extremisers is well-understood yet the identification of their shape has only been established in rather special cases (see, for example, [7], [8], [11], [13], [15]).

In Section 2 we prove Theorem 1.1 and Corollary 1.2, and in Section 3 we adapt our method to obtain an alternative proof of the analogous result from [3] for the case $(d, s) = (5, 1)$.

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2. PROOF OF THEOREM 1.1 AND COROLLARY 1.2

A key ingredient in the proof of Theorem 1.1 is the following sharp inequality proved in [3]. Here we use the notation $y' = \frac{y}{|y|}$, for $y \in \mathbb{R}^d \setminus \{0\}$.

Theorem 2.1. *Let $d \geq 3$. Then*

$$(3) \quad \|e^{it\sqrt{-\Delta}} f\|_{L^4(\mathbb{R}^{d+1})}^4 \leq \mathbf{C}(d) \int_{\mathbb{R}^{2d}} |\widehat{f}(y_1)|^2 |\widehat{f}(y_2)|^2 |y_1|^{\frac{d-1}{2}} |y_2|^{\frac{d-1}{2}} (1 - y'_1 \cdot y'_2)^{\frac{d-3}{2}} dy_1 dy_2$$

holds with sharp constant

$$\mathbf{C}(d) = 2^{-\frac{d-1}{2}} (2\pi)^{-3d+1} |\mathbb{S}^{d-1}|$$

which is attained if and only if

$$\widehat{f}(\xi) = \frac{e^{a|\xi|+b \cdot \xi+c}}{|\xi|},$$

where $a, c \in \mathbb{C}$, $b \in \mathbb{C}^d$ with $|\operatorname{Re}(b)| < -\operatorname{Re}(a)$.

The one-sided wave propagator is given by

$$e^{it\sqrt{-\Delta}} f(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi + it|\xi|} \widehat{f}(\xi) d\xi, \quad (x, t) \in \mathbb{R}^d \times \mathbb{R},$$

for appropriate functions f , and the Fourier transform we use is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-ix \cdot \xi} dx, \quad \xi \in \mathbb{R}^d.$$

Our observation is that if we introduce polar coordinates for y_1 and y_2 in (3), then we are led to real-valued functionals of the form

$$H_\lambda(g) = \int_{\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}} g(\eta_1) \overline{g(\eta_2)} |\eta_1 - \eta_2|^{-\lambda} d\eta_1 d\eta_2$$

for $g \in L^1(\mathbb{S}^{d-1})$ and $\lambda \leq 0$. This is reminiscent of recent work of Foschi [9] where a sharp upper bound for H_{-1} was established for $d = 3$. For Theorem 1.1 we need an analogous result for $d = 4$; this is contained in the subsequent proposition, which we state more generally to highlight why our approach only works as it stands for $d = 4, 5$.

First, we introduce the beta function

$$\mathbf{B}(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \quad x, y > 0,$$

and μ_g to denote the average value of g on the sphere. Also, we use $\mathbf{1}$ for the function which is identically equal to one on the sphere.

Proposition 2.2. *Let $-2 < \lambda < 0$, and let g be any L^1 function on \mathbb{S}^{d-1} . Then,*

$$H_\lambda(g) \leq H_\lambda(\mu_g \mathbf{1}) = 2^{d-2-\lambda} \mathbf{B}\left(\frac{d-1-\lambda}{2}, \frac{d-1}{2}\right) \frac{|\mathbb{S}^{d-2}|}{|\mathbb{S}^{d-1}|} \left| \int_{\mathbb{S}^{d-1}} g \right|^2.$$

Further, equality holds if and only if g is constant.

Following Foschi [9], our proof of Proposition 2.2 is based on a spectral argument using a spherical harmonic decomposition of g and the Funk–Hecke formula to obtain explicit expressions for the eigenvalues. We remark that similar types of arguments have proved profitable in understanding sharp forms of other important estimates; see, for example, [2], [4] and [10]. The connection to the latter paper deserves a further remark; indeed, in [10], Frank and Lieb provide a reproof of the sharp Hardy–Littlewood–Sobolev inequality on the sphere, originally due to Lieb [12], which gives the sharp upper bound on H_λ for $0 < \lambda < d-1$ in terms of the L^p norm of g , where $p = \frac{2(d-1)}{2(d-1)-\lambda}$.

The information we need concerning the eigenvalues is contained in the following lemma. Here we use $P_{k,d}$ to denote the Legendre polynomial of degree k in d dimensions, which may be defined using the generating function

$$\frac{1}{(1+r^2-2rt)^{\frac{d-2}{2}}} = \sum_{k=0}^{\infty} \binom{k+d-3}{d-3} r^k P_{k,d}(t), \quad |r| < 1, |t| \leq 1.$$

Lemma 2.3. *Let $-2 < \lambda < 0$, and define*

$$I_k(d, \lambda) = |\mathbb{S}^{d-2}| \int_{-1}^1 (1-t)^{-\frac{\lambda}{2}} P_{k,d}(t) (1-t^2)^{\frac{d-3}{2}} dt.$$

Then

$$I_0(d, \lambda) = |\mathbb{S}^{d-2}| 2^{d-2-\frac{\lambda}{2}} B\left(\frac{d-1-\lambda}{2}, \frac{d-1}{2}\right) > 0$$

and $I_k(d, \lambda) < 0$ for all $k \geq 1$.

Remark. The inequality in Proposition 2.2 is false if $\lambda < -2$. This is because $(-1)^k I_k(d, \lambda) > 0$ for $k \geq 0$ up to some threshold; for example $I_2(d, \lambda) > 0$ for such λ . This is the reason why our approach does not allow us to prove a generalisation of Theorem 1.1 to dimension 6 and above (for general d , we should take $\lambda = 3 - d$). A similar obstacle arises in [6] when generalising Foschi's argument to obtain the result in [9] in higher dimensions. At the endpoint $\lambda = -2$ the sharp inequality in Proposition 2.2 still holds but one also has equality for certain non-constant functions g . This turns out not to matter for our application and we can recover the sharp inequality and characterisation of extremisers for (1) in the case $(d, s) = (5, 1)$ first proved in [3]; we expand upon this point in Section 3.

Assume Lemma 2.3 to be true for the moment, then to prove Proposition 2.2, we first observe that it suffices by density and continuity of the functional H_λ on $L^1(\mathbb{S}^{d-1})$ to consider $g \in L^2(\mathbb{S}^{d-1})$. We may then write $g = \sum_{k \geq 0} Y_k$ as a sum of orthogonal spherical harmonics; upon which it follows that

$$(4) \quad H_\lambda(g) = 2^{-\frac{\lambda}{2}} \sum_{k \geq 0} \int_{\mathbb{S}^{d-1}} g(\eta_1) \int_{\mathbb{S}^{d-1}} \overline{Y_k(\eta_2)} (1 - \eta_1 \cdot \eta_2)^{-\frac{\lambda}{2}} d\eta_2 d\eta_1.$$

To deal with the inner integral in (4) we use the Funk–Hecke formula for the spherical harmonics, which states that

$$\int_{\mathbb{S}^{d-1}} Y_k(\eta) F(\omega \cdot \eta) d\eta = \Lambda_k Y_k(\omega)$$

for $\omega \in \mathbb{S}^{d-1}$ and $k \in \mathbb{N}_0$, where

$$\Lambda_k := |\mathbb{S}^{d-2}| \int_{-1}^1 F(t) P_{k,d}(t) (1-t^2)^{\frac{d-3}{2}} dt,$$

provided that $F \in L^1([-1, 1], (1-t^2)^{\frac{d-3}{2}})$ (see [1], pp. 35–36). It then follows that the inner integral in (4) evaluates to a (positive) constant multiple of $I_k(d, \lambda) \overline{Y_k(\eta_1)}$. Precisely, using the orthogonality of the spherical harmonics of different degrees and Lemma 2.3,

$$H_\lambda(g) = 2^{-\frac{\lambda}{2}} \sum_{k \geq 0} I_k(d, \lambda) \int_{\mathbb{S}^{d-1}} |Y_k(\eta)|^2 d\eta \leq 2^{-\frac{\lambda}{2}} I_0(d, \lambda) \int_{\mathbb{S}^{d-1}} |Y_0|^2 d\eta = H_\lambda(\mu_g \mathbf{1}).$$

Equality is clearly satisfied for $g = Y_0$ or equivalently g which are constant. There are no further cases of equality since $I_k(d, \lambda)$ is strictly negative for $k \geq 1$, by Lemma 2.3.

Using the expression for $I_0(d, \lambda)$ in Lemma 2.3 and the definition of μ_g , it is then easy to derive the claimed expression for $H_\lambda(\mu_g \mathbf{1})$, which completes the proof of Proposition 2.2.

Proof of Lemma 2.3. By a simple change of variables, it is easily checked that $I_0(d, \lambda)$ satisfies the claimed equality in terms of the beta function. To prove the strict negativity of $I_k(d, \lambda)$ for $k \geq 1$, we first use the Rodrigues formula for $P_{k,d}$ (see [1], pp. 37), which states

$$(1-t^2)^{\frac{d-3}{2}} P_{k,d}(t) = (-1)^k R_{k,d} \frac{d^k}{dt^k} (1-t^2)^{k+\frac{d-3}{2}}, \quad t \in [-1, 1],$$

with

$$R_{k,d} = \frac{\Gamma(\frac{d-1}{2})}{2^k \Gamma(k + \frac{d-1}{2})} > 0,$$

to obtain that

$$I_k(d, \lambda) = (-1)^k R_{k,d} \int_{-1}^1 (1-t)^{-\frac{\lambda}{2}} \frac{d^k}{dt^k} (1-t^2)^{k+\frac{d-3}{2}} dt.$$

Integrating by parts, the boundary terms disappear and we obtain

$$(5) \quad I_k(d, \lambda) = (-1)^k R_{k,d} \left(-\frac{\lambda}{2}\right) \int_{-1}^1 (1-t)^{-\frac{\lambda}{2}-1} \frac{d^{k-1}}{dt^{k-1}} (1-t^2)^{k+\frac{d-3}{2}} dt.$$

Since $-\frac{\lambda}{2} > 0$, the sign of the constant in front of the integral in (5) does not change at the first integration by parts. However, since $-\frac{\lambda}{2} - 1 < 0$, at every integration by parts step after the first, we will incur a sign change. Hence, integrating by parts a total of k times, we see that $I_k(d, \lambda)$ evaluates to

$$-C_k(d, \lambda) \int_{-1}^1 (1-t)^{-\frac{\lambda}{2}-k} (1-t^2)^{k+\frac{d-3}{2}} dt$$

for some strictly positive constant $C_k(d, \lambda)$. Hence $I_k(d, \lambda) < 0$ as claimed. \square

Proof of Theorem 1.1. If we set $d = 4$ and write the integral on the right-hand side of (3) using polar coordinates, we get

$$\int_{(0,\infty)^2} \int_{(\mathbb{S}^3)^2} |\widehat{f}(r_1 \eta_1)|^2 |\widehat{f}(r_2 \eta_2)|^2 r_1^{\frac{9}{2}} r_2^{\frac{9}{2}} (1 - \eta_1 \cdot \eta_2)^{\frac{1}{2}} d\eta dr = \frac{1}{\sqrt{2}} \int_{\mathbb{S}^3 \times \mathbb{S}^3} g(\eta_1) g(\eta_2) |\eta_1 - \eta_2| d\eta_1 d\eta_2,$$

where

$$(6) \quad g(\eta) := \int_0^\infty |\widehat{f}(r\eta)|^2 r^{\frac{9}{2}} dr,$$

for $\eta \in \mathbb{S}^3$. By Plancherel's theorem,

$$\int_{\mathbb{S}^3} g = (2\pi)^4 \|f\|_{\dot{H}^{\frac{3}{4}}(\mathbb{R}^4)}^2.$$

If we then apply (3) and take $\lambda = -1$ in Proposition 2.2, we have

$$(7) \quad \|e^{it\sqrt{-\Delta}} f\|_{L^4(\mathbb{R}^5)}^4 \leq \frac{\mathbf{C}(4)}{\sqrt{2}} H_{-1}(g) \leq \frac{\mathbf{C}(4)}{\sqrt{2}} H_{-1}(\mathbf{1}) |\mu_g|^2 = \frac{4}{15\pi^2} \|f\|_{\dot{H}^{\frac{3}{4}}(\mathbb{R}^4)}^4,$$

as claimed. The first inequality in (7) is an equality when f extremises inequality (3), and the second is an equality when the function g defined by (6) is constant on \mathbb{S}^3 . In particular, equality holds in both cases for f given by

$$\widehat{f}(\xi) = \frac{e^{a|\xi|+ib\cdot\xi+c}}{|\xi|},$$

where $a, c \in \mathbb{C}$ such that $\operatorname{Re}(a) < 0$, and $b \in \mathbb{R}^4$. Note that, for such f , we have that $|\widehat{f}|$ is radial (and hence g is constant).

On the other hand, if f is an extremiser for (2), then we must have equality at both of the inequalities in (7). From the first inequality, using Theorem 2.1, we see that necessarily

$$\widehat{f}(\xi) = \frac{e^{a|\xi|+b\cdot\xi+c}}{|\xi|},$$

where $a, c \in \mathbb{C}$, $b \in \mathbb{C}^4$ and $\operatorname{Re}(a) < -|\operatorname{Re}(b)|$. However, in this case,

$$g(\eta) = e^{2\operatorname{Re}(c)} \int_0^\infty e^{2r(\operatorname{Re}(a)+\operatorname{Re}(b)\cdot\eta)} r^{\frac{5}{2}} dr = \frac{e^{2\operatorname{Re}(c)} \int_0^\infty e^{-r} r^{\frac{5}{2}} dr}{[-2(\operatorname{Re}(a) + \operatorname{Re}(b) \cdot \eta)]^{\frac{7}{2}}}$$

and for this to be constant in η , we must have $\operatorname{Re}(b) = 0$. This completes the proof of Theorem 1.1. \square

Proof of Corollary 1.2. Write the solution of the wave equation u as $e^{it\sqrt{-\Delta}}f_+ + e^{-it\sqrt{-\Delta}}f_-$, where the functions f_+ and f_- are defined using the initial data by

$$u(0) = f_+ + f_-, \quad \partial_t u(0) = i\sqrt{-\Delta}(f_+ - f_-).$$

Using orthogonality and the Cauchy–Schwarz inequality on $L^2(\mathbb{R}^5)$, we get

$$\begin{aligned} \|u\|_{L^4(\mathbb{R}^5)}^4 &= \|e^{it\sqrt{-\Delta}}f_+\|_{L^4(\mathbb{R}^5)}^4 + \|e^{-it\sqrt{-\Delta}}f_-\|_{L^4(\mathbb{R}^5)}^4 + 4\|e^{it\sqrt{-\Delta}}f_+e^{-it\sqrt{-\Delta}}f_-\|_{L^2(\mathbb{R}^5)}^2 \\ &\leq \|e^{it\sqrt{-\Delta}}f_+\|_{L^4(\mathbb{R}^5)}^4 + \|e^{-it\sqrt{-\Delta}}f_-\|_{L^4(\mathbb{R}^5)}^4 + 4\|e^{it\sqrt{-\Delta}}f_+\|_{L^4(\mathbb{R}^5)}^2 \|e^{-it\sqrt{-\Delta}}f_-\|_{L^4(\mathbb{R}^5)}^2. \end{aligned}$$

The basic inequality $2(X^2 + Y^2 + 4XY) \leq 3(X + Y)^2$ and Theorem 1.1, which clearly also holds for $e^{-it\sqrt{-\Delta}}$, now yield

$$\|u\|_{L^4(\mathbb{R}^5)}^4 \leq \frac{3}{8} \mathbf{W}(4, \frac{3}{4})^4 \left(\|u(0)\|_{\dot{H}^{\frac{3}{4}}(\mathbb{R}^4)}^2 + \|\partial_t u(0)\|_{\dot{H}^{-\frac{1}{4}}(\mathbb{R}^4)}^2 \right)^2$$

which gives the claimed inequality in Corollary 1.2.

The above argument was used by Foschi in [8] when $(d, s) = (3, \frac{1}{2})$ and in [3] when $(d, s) = (5, 1)$. The characterisation of extremisers also follows in the analogous way, and so we refer the reader to [8] or [3] and omit the details. \square

3. FIVE SPATIAL DIMENSIONS

We conclude by presenting an alternative derivation of the sharp constant and characterisation of extremisers for the estimate (1) in the case $(d, s) = (5, 1)$, in the spirit of the argument in the previous section.

For this, we need an appropriate modification of Proposition 2.2 and thus Lemma 2.3 for $d = 5$ and $\lambda = -2$. However, it is straightforward to see that

$$I_k(5, -2) = |\mathbb{S}^3| \int_{-1}^1 (1-t) P_{k,5}(t) (1-t^2) dt$$

satisfies $I_0(5, -2) > 0$, $I_1(5, -2) < 0$ and $I_k(5, -2)$ vanishes for all $k \geq 2$. Thus

$$H_{-2}(g) = \frac{I_0(5, -2)}{2} \|Y_0\|_{L^2(\mathbb{S}^4)}^2 + \frac{I_1(5, -2)}{2} \|Y_1\|_{L^2(\mathbb{S}^4)}^2 \leq H_{-2}(\mathbf{1}) |\mu_g|^2,$$

where $g = \sum_{k \geq 0} Y_k$ is the expansion of g into spherical harmonics. Here, equality holds if g is constant, but unlike the estimates in Proposition 2.2, there are further cases of equality.

Taking $f \in \dot{H}^1(\mathbb{R}^5)$ and applying this with g given by

$$(8) \quad g(\eta) := \int_0^\infty |\widehat{f}(r\eta)|^2 r^6 dr$$

for $\eta \in \mathbb{S}^4$, we have

$$\|e^{it\sqrt{-\Delta}}f\|_{L^4(\mathbb{R}^6)}^4 \leq \frac{\mathbf{C}(5)}{2} H_{-2}(g) \leq \frac{\mathbf{C}(5)}{2} H_{-2}(\mathbf{1}) |\mu_g|^2 = \frac{1}{24\pi^2} \|f\|_{\dot{H}^1(\mathbb{R}^5)}^4.$$

As before, equality holds in both inequalities for f given by

$$\widehat{f}(\xi) = \frac{e^{a|\xi|+ib\cdot\xi+c}}{|\xi|},$$

where $a, c \in \mathbb{C}$ such that $\operatorname{Re}(a) < 0$, and $b \in \mathbb{R}^5$.

Conversely, if f is an extremiser, then Theorem 2.1 implies that

$$\widehat{f}(\xi) = \frac{e^{a|\xi|+b\cdot\xi+c}}{|\xi|},$$

where $a, c \in \mathbb{C}$, $b \in \mathbb{C}^5$ and $\operatorname{Re}(a) < -|\operatorname{Re}(b)|$. Substituting our function f into (8), we see that it suffices to consider

$$g(\eta) = e^{2\operatorname{Re}(c)} \int_0^\infty e^{2r(\operatorname{Re}(a)+\operatorname{Re}(b)\cdot\eta)} r^4 dr = \frac{e^{2\operatorname{Re}(c)}}{32(-\operatorname{Re}(a) - \operatorname{Re}(b)\cdot\eta)^5} \int_0^\infty e^{-r} r^4 dr.$$

Since $I_1(5, -2) < 0$, we must have that $\|Y_1\|_{L^2} = 0$. On the other hand, using the projection Π onto the space of spherical harmonics of degree one given by

$$\Pi g(\eta) \mapsto \frac{5}{|\mathbb{S}^4|} \int_{\mathbb{S}^4} P_{1,5}(\eta \cdot \omega) g(\omega) d\omega$$

for each $\eta \in \mathbb{S}^4$, it follows that

$$(9) \quad Y_1(\eta) = C \int_{\mathbb{S}^4} \frac{P_{1,5}(\eta \cdot \omega)}{(-\operatorname{Re}(a) - \operatorname{Re}(b)\cdot\eta)^5} d\omega$$

for some absolute constant $C > 0$. If we suppose, for a contradiction, that $\operatorname{Re}(b) \neq 0$, then an application of the Funk–Hecke formula implies that

$$(10) \quad Y_1(\eta) = C P_{1,5}(\eta \cdot \operatorname{Re}(b)') \int_{-1}^1 \frac{t(1-t^2)}{(1+At)^5} dt$$

for each $\eta \in \mathbb{S}^4$, where $A := \frac{|\operatorname{Re}(b)|}{\operatorname{Re}(a)} \in (-1, 0]$. The absolute constants $C > 0$ in (9) and (10) may not be the same. Since Y_1 vanishes almost everywhere on \mathbb{S}^4 , it follows that the integral on the right-hand side of (10) vanishes. This forces $A = 0$, which gives the desired contradiction.

The above argument provides an alternative proof of the following, and at the level of the proof, unifies it with Theorem 1.1.

Theorem 3.1 ([3], Corollary 2.2). *The one-sided wave propagator satisfies the estimate*

$$\|e^{it\sqrt{-\Delta}}f\|_{L^4(\mathbb{R}^6)} \leq \mathbf{W}(5, 1) \|f\|_{\dot{H}^1(\mathbb{R}^5)}$$

with constant

$$\mathbf{W}(5, 1) = \left(\frac{1}{24\pi^2} \right)^{\frac{1}{4}}.$$

The constant is sharp and is attained if and only if

$$\widehat{f}(\xi) = \frac{e^{a|\xi|+ib\cdot\xi+c}}{|\xi|},$$

where $a, c \in \mathbb{C}$ such that $\operatorname{Re}(a) < 0$, and $b \in \mathbb{R}^5$.

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